Zoom Birational Geometry Seminar (01/20/2023)

Finite order birational automorphisms of Fano hypersurfaces

(with Nathan Chen and David Stapleton)

§1. Intro / $\mathbb{C}$ $\quad X = X_d \subseteq \mathbb{P}^{n+1}_{\mathbb{C}}$ smooth hypersurface of deg d $(n \geq 2)$

Q1. What are the automorphisms of X?

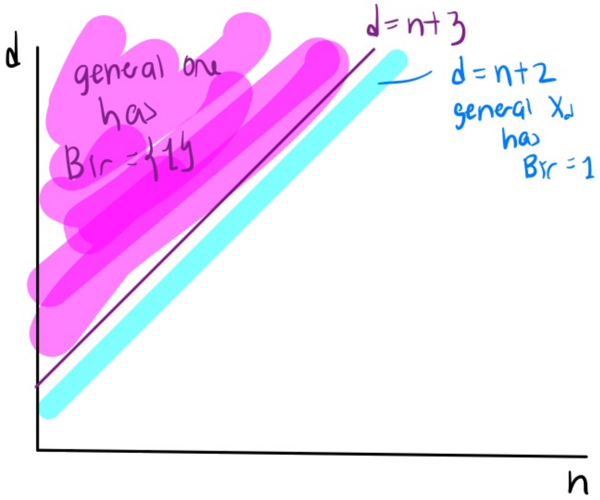
Thm (Matsumura–Monsky '63).

If $d \geq 3$ and $(n, d) \neq (2,4)$, then $\mathrm{Aut}(X) = \mathrm{Lin}(X)$ ← induced by $\mathrm{Aut}(\mathbb{P}^{n+1})$

and if X is general, $\mathrm{Aut}(X) = \{1\}$.

Q2. What about $\mathrm{Bir}(X)$? ← self-maps $X \overset{bir}{\dashrightarrow} X$

Depends on $K_X = \mathcal{O}_X(-n-2+d)$



$d \geq n+3$: (K_X positive case): X general type

Thm (Matsumura '63)

For Y variety of general type, $\mathrm{Aut}(Y) = \mathrm{Bir}(Y) = $ finite.

(For X_d: $\mathrm{Bir}(X_d) = \mathrm{Aut}(X_d) = \{1\}$ for X_d general)

$d = n+2$: $K_X = 0$ $\quad X$ Calabi-Yau

Fact (MMP): If Y smooth proj variety with $K_Y \cong \mathcal{O}_Y$ and $\rho(Y) = 1$, then $\mathrm{Bir}(Y) = \mathrm{Aut}(Y)$.

Q: What about the Fano range?

One extreme: $d=1$: $\mathrm{Bir}(\mathbb{P}^n) = Cr_n(\mathbb{C})$. This is huge and mysterious.

$n=2$: Cremona thm: $Cr_2(\mathbb{C})$ is gen by PGL_3 and involutions

Thm (Lin–Shinder '22)

$Cr_n(\mathbb{C})$ $(n \geq 4)$ is not gen by pseudo-regularizable elements.

Thm (Blanc–Schneider–Yasinsky '22)

For $n \geq 4$, $Cr_n(\mathbb{C}) \twoheadrightarrow$ free group on $\mathbb{C}$

Thm (BSY) For any complex variety Y

$\exists$ surj. $Cr_n(\mathbb{C}) \twoheadrightarrow \mathrm{Bir}\, Y$

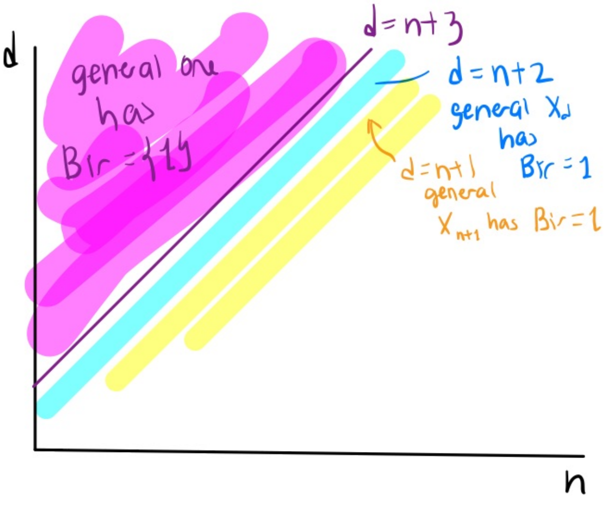
Other extreme for Fanos:

$d = n+1$: (Fano Index 1: $K_X = \mathcal{O}(-1)$)

Thm (Fano–Segre–Iskovskikh–Manin–Pukhlikov–Corti–Cheltsov–de Fernex–Ein–Mustaţă–Zhuang)

If $n \geq 3$, every smooth $X_{n+1} \subseteq \mathbb{P}^{n+1}$ has $\mathrm{Bir} = \mathrm{Aut}$

$\Rightarrow$ for general one, $\mathrm{Bir} = 1$



$d = n$ (Fano index 2)

Thm (Pukhlikov 2016)

If $n \geq 14$, then a general $X_n \subseteq \mathbb{P}^{n+1}$ has $\mathrm{Bir} = \mathrm{Aut} = 1$

Higher index Fanos?

Thm A (Chen–J.–Stapleton) p prime, n, d integers s.t. $d \geq p\left\lceil \frac{n+3}{p+1} \right\rceil$

Assume n is even if $p = 2$. Then for very general $X_d \subseteq \mathbb{P}^{n+1}_{\mathbb{C}}$, any finite order element in $\mathrm{Bir}(X_d)$ has order a p-power.

Cor B $X_d \subseteq \mathbb{P}^{n+1}_{\mathbb{C}}$. If i) $d \geq 3\left\lceil \frac{n+3}{4} \right\rceil$ and n is even, $\leftarrow$ $p = 2, 3$

or ii) $d \geq 5\left\lceil \frac{n+3}{6} \right\rceil$ and n is odd $\leftarrow$ $p = 3, 5$

then $\mathrm{Bir}(X_d)$ has no finite order elements.

Parity assumption:

comes from studying double covers of hypersurfaces in char 2

n odd case: don't have an explicit resol. at the moment

§2. Ingredients in proof

R DVR, π uniformizer, $\underset{\eta}{K = \mathrm{Frac}}$, $\underset{0}{k = R/\pi}$

3 main ingredients

1) Mori's construction

$$\begin{tikzcd}\mathcal{X} \arrow[d] \\ \mathrm{Spec}\, R\end{tikzcd}$$

$\mathcal{X}_\eta \subseteq \mathbb{P}^{n+1}_K$ hypersurface

$\mathcal{X}_0$ = deg p cover of a hypersurface $/k$

2) Thm (Chen–Stapleton) If R has mixed char$(0,p)$ and p, n, d are as in Thm A, then $\mathrm{Bir}(\mathcal{X}_0) = \{1\}$.

3) Specialization map $sp_\eta : \mathrm{Bir}(\mathcal{X}_\eta) \longrightarrow \mathrm{Bir}(\mathcal{X}_0)$

We show that $\ker(sp_\eta)$ contains no finite order elements of order coprime to $\mathrm{char}(k)$.

Mori's construction:

deg p cover of hypersurface

$$\begin{tikzcd}\mathcal{X}_\eta \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] & \mathcal{X}_0 \arrow[l, hook'] \arrow[d] \\ \mathrm{Spec}\, K \arrow[r] & \mathrm{Spec}\, R & \mathrm{Spec}\, k \arrow[l, hook']\end{tikzcd}$$

hypersurface

Kollár: observed for certain choices of n, degrees, p,

$\Downarrow$

$\mathcal{X}_0$ carries unusual diff forms in char p

Thm (Kollár): $X_d \subseteq \mathbb{P}^{n+1}$ very gen $d \geq 2\lceil \frac{n+3}{3} \rceil$

$\Rightarrow X_d$ not ruled.

Totaro, Schneider also used this degeneration.

How the ingredients $\Rightarrow$ Thm A.

Take Mori's construction for n, p, d in Thm A

deg d hypersurface

deg p cover in char p

$$\begin{tikzcd}\mathcal{X}_\eta \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] & \mathcal{X}_0 \arrow[l, hook'] \arrow[d] \\ \mathrm{Spec}\, K \arrow[r] & \mathrm{Spec}\, R & \mathrm{Spec}\, k \arrow[l, hook']\end{tikzcd}$$

mixed char $(0,p)$

For Bir:

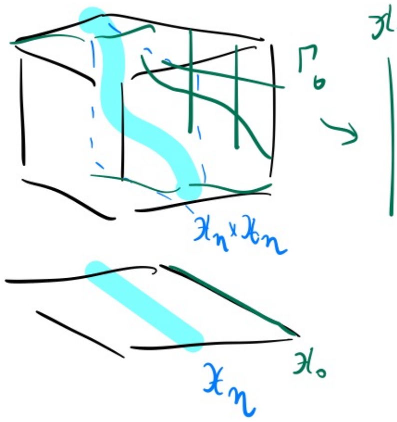
$$sp_{\bar\eta} : \mathrm{Bir}(\mathcal{X}_{\bar\eta}) \longrightarrow \mathrm{Bir}(\mathcal{X}_{\bar 0}) = \{1\}$$

$\uparrow$ Chen-Stapleton

$\ker(sp_{\bar\eta})$ can't contain finite order elements of order coprime to char k

$\Rightarrow \mathrm{Bir}(\mathcal{X}_{\bar\eta})$ has no finite order elements of order coprime to p

§3. Specialization for Bir First appeared in Matsusaka–Mumford '64



$\mathcal{X}_R$ normal + integral, separated + flat / R
Assume $\mathcal{X}_0$ is integral.

For $\varphi \in \mathrm{Bir}(\mathcal{X}_\eta)$, let $\Gamma \subset \mathcal{X}_R \times \mathcal{X}_R$ be the closure of the graph.

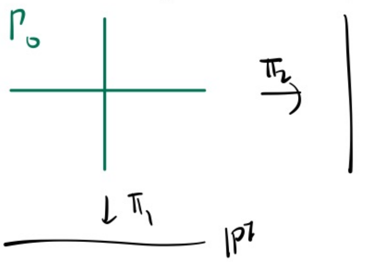
φ <u>specializes to $\mathcal{X}_0$</u> if Γ_0 has a unique irred component that maps birationally to $\mathcal{X}_0$ under both projections.

In general, not every $\varphi \in \mathrm{Bir}(\mathcal{X}_\eta)$ specializes well:

<u>Exmp</u>: $\mathcal{X}_R = \mathbb{P}^1_x \times \mathbb{A}^1_t$ Spec $R = \mathbb{A}^1_t$

$\varphi: x \mapsto \dfrac{t}{x}$

does not specialize to $\mathcal{X}_0$



In the non-(uni)ruled setting, specialization behaves well

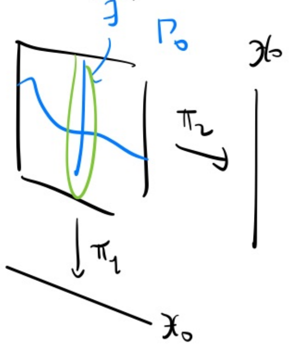
<u>Facts</u>: 1) $\{\varphi \in \mathrm{Bir}(\mathcal{X}_\eta)$ that specialize to $\mathcal{X}_0\}$ is a subgroup of $\mathrm{Bir}(\mathcal{X}_\eta)$

2) If $\mathcal{X}_R$ is proper and has (separably uni-) ruled modifications, and $\mathcal{X}_0$ is not (separably uni-) ruled, then every $\varphi \in \mathrm{Bir}(\mathcal{X}_\eta)$ specializes.
$\Rightarrow$ get a group homom $\mathrm{sp}_\eta: \mathrm{Bir}(\mathcal{X}_\eta) \to \mathrm{Bir}(\mathcal{X}_0)$

<u>Defn</u>: Y has (separably uni-) ruled modifications if every exc divisor of every normal birat modification is (separably uni-) ruled.

<u>Exmp</u>: 1) <u>Abhyankar</u>: Y regular $\Rightarrow$ has ruled modification
2) <u>Chen-Stapleton</u>: If $\mathcal{X}_R$ is Mori's construction / $R = \mathbb{Z}_p^{sh}$ (and n even if p=2) then $\mathcal{X}_R$ has sustained separably uniruled modification

<u>Why?</u> Take $\varphi \in \mathrm{Bir}(\mathcal{X}_\eta)$, Γ = closure of graph
look at Γ_0:
If component gets contracted by π_1, get something exceptional
Assumption $\rightsquigarrow$ component is ruled



$\Rightarrow\Leftarrow$

Prop: Let $\ell > 1$, $\varphi \in \mathrm{Bir}(\mathcal{X}_\eta)$ an element of order ℓ that specializes to $\mathcal{X}_0$.
If ℓ is invertibu in R, then $sp_\eta(\varphi)$ has order ℓ.
In particular, $\varphi \notin \ker(sp_\eta)$.

Sketch: Idea: Regularize it, take the quotient. Use projection formula. "$\square$"
(φ is defined on $U \rightsquigarrow$ take $\bigcap_{i=1}^{\ell} \varphi^i(U)$)

§4. Kernel of specialization

⚠ sp_η is not injective in general

Exmp1: Take $\mathcal{X}_R \leftarrow$ family of deg p coves of a hypersurfau
$\downarrow$
$R \leftarrow$ mixed char $(0,p)$

has order p covering automophiim
$\mathcal{X}_\eta$ = p-cyclic cover
$\mathcal{X}_0$ = deg p cover satisfying numerics in Chen–Stapleton
has $\mathrm{Bir} = \{1\}$

Example where $\ker(sp_\eta)$ contains p-torsion.

Exmp 2: (Family of K3 surfaces where $\ker(sp_\eta)$ contains infinite order elements)

$X_t = (1,1) \cap (2,2) \in \mathbb{P}^2 \times \mathbb{P}^2$

$\pi_i : X_t \to \mathbb{P}^2$ induce covering involutions $\iota_i : X_t \to X_t$

For X_t general, ι_1 and ι_2 don't commute

$$\mathrm{Bir}(X_t) = \underset{\iota_1}{(\mathbb{Z}/2)} * \underset{\iota_2}{(\mathbb{Z}/2)}$$

For special X_0, ι_1 and ι_2 do commute.

$\mathcal{X}$
$\downarrow$
R

and $\varphi = \iota_1 \iota_2 \iota_1 \iota_2$ infinite order element in $\mathrm{Bir}(\mathcal{X}_\eta)$ in kernel of sp_η

Exmp: (Dolgachev).

S K3
$\cap$
$\mathbb{P}^4$

$S^{[2]} \longrightarrow S^{[2]}$

$(p_1, p_2) \longmapsto (q_1, q_2)$

where $\ell_{p_1 p_2} \cap S = p_1 + p_2 + q_1 + q_2$

$$\begin{tikzcd} \mathrm{Aut}(\mathcal{X}_\eta) \arrow[r, "/"] \arrow[dr] & \mathrm{Aut}(\mathcal{X}_0) \arrow[d, phantom, "\cap"] \\ & \mathrm{Bir}(\mathcal{X}_0) \end{tikzcd}$$

Lieblich–Maulik: Cone conj for K3 in positive char.

If $sp : \mathrm{Pic}(\mathcal{X}_{\bar\eta}) \xrightarrow{\cong} \mathrm{Pic}(\mathcal{X}_0)$ and $\mathcal{X}_R \to R$ is a family of K3s, then $sp_{\bar\eta}$ is injective.